Boundary spatiotemporal correlations in a self-organized critical model of punctuated equilibrium

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In a semi-infinite geometry, a one-dimensional, M-component model of biological evolution realizes microscopically an inhomogeneous branching process for $M\rightarrow\infty$. This implies a size distribution exponent $\tau'=7/4$ for avalanches starting at a free, ''dissipative'' end of the evolutionary chain. A bulklike behavior with $\tau'=3/2$ is restored by ''conservative'' boundary conditions. These are such as to strictly fix to its critical, bulk value the average number of species directly involved in an evolutionary avalanche by the mutating species located at the chain end. A two-site correlation function exponent $\tau_R{}'=4$ is also calculated exactly in the ''dissipative'' case, when one of the points is at the border. Together with accurate numerical determinations of the time recurrence exponent τ'_{first}, these results show also that, no matter whether dissipation is present or not, boundary avalanches have the same space and time fractal dimensions as those in the bulk, and their distribution exponents obey the basic scaling laws holding there.

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I. INTRODUCTION

Nature offers many examples of systems driven by some external force towards an out-of-equilibrium state characterized by critical spatiotemporal correlations [1]. In this stationary state the accumulated stress is dissipated by avalanches of activity which occur intermittently and cover all spatial and temporal scales. Models of nonequilibrium critical dynamics displaying such features have been proposed for several phenomena, ranging from earthquakes [2] to interface depinning [3], or biological evolution in ecosystems [4].

Some models of self-organized criticality (SOC) are characterized by extremal dynamics. Among these, the model of biological evolution introduced by Bak and Sneppen (BS) constitutes an important example [4]. Especially for a system with extremal dynamics, very few exact results are available so far [5]. Most of our insight is based on numerical simulations, scaling arguments [6], or mean field solutions, related in general to random neighbor versions of the models [7].

Among the existing mean field approaches, a particularly rich and complete one, proposed recently, allows several properties of avalanches, including some due to border effects, to be described in terms of an inhomogeneous branching process (BP) [8]. In view of its phenomenological character, an open interesting problem within such an approach remains the identification of specific microscopic models realizing the scalings of the inhomogeneous BP in some appropriate random neighbor or similar limit [9].

A step towards establishing an analytical theory of scaling in extremal dynamics systems has been taken recently by Boettcher and Paczuski [10], who computed exactly a correlation function of an M-component version of the BS evolution model in the limit when M approaches infinity. This result, combined with numerical ones, allowed important general scaling laws for self-organized critical behavior [6] to be verified. Such laws describe the connection between space and time fractal properties of avalanches in the bulk.

For models like sandpiles, SOC can be established by the effects of boundary dissipation, which balances the flux of added particles [1]. This basic circumstance, together with experience with standard criticality, called recent attention to the boundary scaling properties of avalanches [11,12]. Surface scaling in sandpiles can be different from bulk scaling and can also depend on the type of boundary conditions (b.c.'s) considered. In sandpile models the obvious alternative to dissipative b.c's is conditions in which part of the border does not dissipate grains [11]. For evolution models, which do not dissipate particles, it is not known whether boundary conditions could influence scaling at the border and, if so, what should correspond to conservation. These are issues we address in the present article.

Especially in the perspective of obtaining exact results, the study of boundary scaling should represent an important step towards a deeper and more complete theoretical understanding of SOC. In the present article, we show that the $M=\infty$ model of Ref. [10], if considered in the presence of boundaries, provides a microscopic realization of the inhomogeneous BP introduced in [8]. Besides immediately generalizing results known for the BP to this model, this opens new possibilities for both analytical and numerical investigations. In particular, by extending methods used previously for the bulk [10], we are able to compute exactly the asymptotic two-point correlator when it involves a point on the boundary in the $M=\infty$ limit. These results, together with an accurate numerical analysis of time correlations at the boundary, allow a complete scenario of the scalings obeyed by boundary and bulk exponents of the system to be drawn. Boundary avalanches have different scaling properties for different boundary conditions. However, even in the case of a dissipative border, in which exponents differ from those in the bulk, space and time fractal dimensions of the avalanches remain unaltered and satisfy the same basic scaling relations.

This paper is organized as follows. In the next section, after introducing the semi-infinite M-component model, we discuss its relation to an inhomogeneous BP in the $M\to\infty$ limit and derive a number of analytical and numerical results for the properties of the size and spatial distribution. The third section is devoted to a discussion of exponents related to the time recurrence of activity at the border. The last section contains the conclusions.

II. SEMI-INFINITE BS EVOLUTION CHAIN WITH $M=\infty$ COMPONENTS

The possible relevance of the BS model for evolution, as revealed, e.g., by paleontological records [13], has been already discussed in the literature. Here we regard this model and its variants as an interesting mathematical framework within which SOC dynamics can be studied.

We consider an open chain of species, labeled by an index $i=1,2,3\ldots$. Each species is characterized by M independent parameters (traits) x_i^{α} ($\alpha=1,2,\ldots,M$, $0<x_i^{\alpha}<1$), which quantify the ability of the species to survive in connection with M different tasks it has to perform in the ecosystem. The closer x_i^{α} is to 1, the higher the ability connected to the αth task, and thus the greater the chance that the species avoids mutation.

The dynamics goes as follows. At every time step the smallest x_i^{α}, i.e., the weakest among all the traits of all species, is identified and replaced by 1. Each one of the species which are neighbors along the chain of that site, i_{min}, with minimum x^{α}, get one of their traits (chosen at random among the M possible ones) replaced by new random numbers extracted independently and with uniform probability in the interval (0,1). A new minimum is then searched for and this proceeds so that at long times the system self-organizes itself into a stationary state with all x_i^{α} uniformly distributed in an interval $(\lambda_c,1)$.

A λ avalanche is identified with a sequence of mutations starting at site i_{min} with $x_{i_{min}}^{\alpha}=\lambda$, and continuing until the current minimum x^{α} remains below λ. We call the total number of minima with value below λ obtained during the avalanche, s (the duration of the avalanche).

Rather then considering a translationally invariant situation as in Ref. [10], we take here a semi-infinite chain, with a suitable b.c. The probability that in the stationary state a λ avalanche has size s will thus depend on the site j ($j=1,2,\ldots$), where the avalanche started. Omitting the λ dependence, we indicate such probability by $P_j(s)$. The above dynamical rules, for $M\to\infty$, lead to

$$P_j(s+1)=\lambda(1-\lambda)[P_{j+1}(s)+P_{j-1}(s)]+\lambda^2\sum_{s'=0}^{s}P_{j+1}(s')P_{j-1}(s-s'),\qquad j>1. \tag{1}$$

Equation (1) is derived on the basis of the same considerations as those made made in Ref. [10]. In first place Eq. (1) reflects the fact that, with our dynamical rules, the starting active site (site j for which $x_j^{\alpha}=\lambda=$absolute minimum) can either activate one of its neighbors with probability $\lambda(1-\lambda)$ or both of them with probability λ^2. In the two cases, of course, one or two independent avalanches follow, respectively, and the global avalanche grows up to a total of $s+1$ activated sites. The above independence, which allows Eq. (1) to be written in such a form, holds in the $M\to\infty$ limit, which is implicitly assumed here. Indeed, only in this limit is the evolution of an avalanche completely unaffected by the fact that a given site has been previously involved in the same, or another, avalanche. The effects of this kind of condition can indeed be seen to amount to corrections of the order $1/M$, or higher, in the equations of motion.

The b.c.'s complementing Eq. (1) can be written in different ways. A first possibility is

$$P_1(s+1)=\lambda P_2(s). \tag{2}$$

This means that, once the boundary site 1 becomes active, it can then activate only site 2 (site 0 does not exist), and this occurs with the usual rules as in the bulk. An alternative boundary condition is

$$P_1(s+1)=\lambda(1-\lambda)[P_1(s)+P_2(s)]+\lambda^2\sum_{s'=0}^{s}P_1(s')P_2(s-s'), \tag{3}$$

which means that for the boundary site, when active, the rule of setting $x^{\alpha}=1$, valid for $j>1$, does not apply. On the contrary, $j=1$ and $j=2$ get now the random replacement of one of their traits, as sites $j-1$ and $j+1$ in the bulk [see Eq. (1)]. In other words, the role of the missing site $j=0$ is now played by the site $j=1$ itself.

It is straightforward to recognize that, up to minor modifications due to the convention assumed here of replacing the x associated with the minimum trait by 1, Eqs. (1) and (2) have the same structure as those describing the inhomogeneous BP in one dimension (1D) of Ref. [8]. By introducing generating functions $\tilde{P}_i(x)=\Sigma_{s=0}^{\infty}P_i(s)x^s$, $i=1,2,\ldots$, it was found there that $\tilde{P}_1(x)\sim 1+c(1-x)^{1-7/4}$, for $x\to 1^-$, when $\lambda=1/2$. This value of λ implies an average number $2\lambda(1-\lambda)+2\lambda^2=1$ of sites activated by each active site in the bulk and coincides with λ_c [10]. The average number of sites activated by the border site $i=1$ is instead less than unity, according to Eq. (2). By applying the methods of Ref. [8], one can show easily that, for $\lambda=\lambda_c=1/2$, $P_1\sim s^{-\tau'}$, with $\tau'=7/4$, when $s\to\infty$. This result holds independent of the convention assumed here of replacing the minimum trait by unity. The slightly different equations of Ref. [8] reflect the fact that there also the minimum x^{α} was replaced by a new random number. The asymptotic behavior of P_1 has to be compared with the result $P_{\infty}\sim s^{-3/2}$ [10] holding when the site where the avalanche starts is chosen in the bulk, and implying a mean field bulk exponent $\tau=3/2$ [7].

We indicate by $N(j,r)$ the probability that, at $\lambda=\lambda_c=1/2$, an avalanche started at site j never reaches site $r\geqslant j$. We are interested in the behavior of N for $j=1$ and large r, which is in turn related to the asymptotic radial distribution of avalanches starting at the border of the chain. The Markovian nature of avalanche evolution leads to

$$N(j,r)=\frac{1}{4}+\frac{1}{4}[N(j+1,r)+N(j-1,r)]+\frac{1}{4}N(j+1,r)N(j-1,r), \tag{4}$$

for $2<j<r-1$. The b.c. in Eq. (2) implies

$$N(1,r)=\frac{1}{2}+\frac{1}{2}N(2,r), \tag{5}$$

while, since obviously $N(r,r)=0$,

$$N(r-1,r)=\frac{1}{4}+\frac{1}{4}N(r-2,r). \tag{6}$$

Next, we put $N(j,r)=1-f(j,r)$, so that f represents the probability that an avalanche starting in site j reaches site r. From the above equations, it follows that

$$\Delta f(j,r)=\frac{1}{2}f(j-1,r)f(j+1,r) \qquad 1<j<r-1,$$

$$\Delta f(r-1,r)=\frac{1}{2}-\frac{3}{4}f(r-2,r),$$

$$\Delta f(1,r)=\frac{1}{2}f(2,r), \tag{7}$$

where $\Delta f(k,r)$ is the discrete Laplacian of f at site k. Since we are interested in the large r behavior, we can pass to a continuum limit, introducing the variable $z=(j-1)/r$. By putting $f(j,r)=y(z)$, we obtain from Eq. (7)

$$\frac{y''(z)}{r}=\frac{y(z)}{2}, \quad 0<z<1, \tag{8}$$

with boundary conditions $y'(1)/r=-3y(1)/4+1/2$ and $y'(0)/r=y(0)/2$. Equation (8) can be integrated and, after some algebra, one finds

$$f(1,r)=y(0)\simeq\frac{6}{r^3}\left[\int_0^\infty\frac{dx}{\sqrt{x^3+1}}\right]^3 \simeq\frac{2}{9}\frac{1}{r^3}\left[\frac{\Gamma(1/3)\Gamma(1/6)}{\Gamma(1/2)}\right]^3\simeq\frac{20.14\ldots}{r^3}. \tag{9}$$

If the probability that a critical avalanche starting at site 1 reaches site r is $P_{R1}(r)\propto r^{-\tau'_R}$, we conclude that $\tau'_R=4$, from the fact that $f(1,r)\propto\int_r^\infty P_{R1}(x)dx$. The above derivation extends the approach of Ref. [10], which yielded $\tau_R=3$ for the bulk $P_R(r)$.

The presence of a border like that specified by the b.c. in Eq. (2) would be expected to make avalanche propagation more difficult as compared to the bulk situation. Thus, the result $\tau'_R=4$ is physically sound compared to $\tau_R=3$. The fact that the radial probability distribution function decays with a different exponent when the starting point is at the border with b.c. (2) is qualitatively consistent with what we know of the two-point correlator at an equilibrium critical point when one of the points is fixed at the boundary and not in the bulk [14].

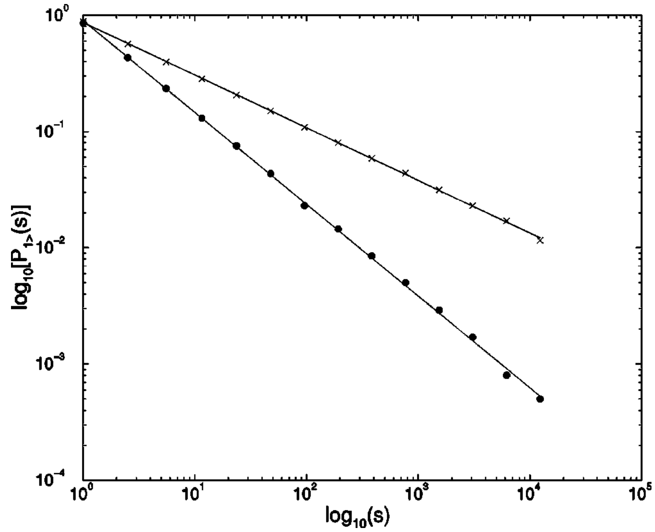

FIG. 1. The log-log plots of the integrated s distribution of avalanches starting at site 1, $P_{1>}(s)=\int_s^\infty P(x)dx$ with b.c. (2) (dots) and b.c. (3) (crosses).

The exact results above for τ' and τ'_R allow us to draw a first conclusion on the space fractal dimension, D', of avalanches starting at a border with b.c. (2). Assuming $s\propto r^{D'}$ for such an avalanche leads to $D'(1-\tau')=1-\tau'_R$, which follows from $P_1(s)ds=P_{R1}(r)dr$. Thus, $\tau'=7/4$ and $\tau'_R=4$ imply $D'=4$. This dimension coincides with the bulk one, D, which satisfies the same kind of relation $D(1-\tau)=1-\tau_R$ [10]. So, at the boundary, there is no distinct space fractal dimension for these avalanches, in spite of the different τ exponent.

We verified the above result for τ' numerically, by simulating the model on open finite chains of length $N=200$, with $M=100$ components. Figure 1 reports our finite-size data referring to avalanches starting at the border with b.c.'s given by Eq. (2). The distribution is in good accord with the expected $\tau'=7/4$ (we estimated $\tau'=1.78\pm0.04$). Direct simulation allows us also to investigate the implications of b.c. (3), which, as far as τ' is concerned, cannot be dealt with analytically. For these b.c.'s, which keep the average number of sites activated by the border site equal to 1, we find numerically $\tau'=1.46\pm0.04$, compatible with $\tau'=\tau=3/2$ (Fig. 1). Since the result $\tau'=7/4$ should hold for the inhomogeneous BP as long as $2\lambda(1-\lambda)+\lambda^2<1$ [8], we conclude that $\tau'=\tau=3/2$ is peculiar to b.c. (3). Consistently one can also show that, with b.c. (3), $\tau'_R=\tau_R=3$ exactly. Thus, also τ'_R is restored to its bulk value by b.c. (3). In the SOC context, similar results were previously conjectured, on a numerical basis, for the Abelian sandpile in two dimensions [11]. Indeed, for that model border avalanches appeared to possess a toppling distribution exponent rather close to the bulk value for a conservative border, while with boundary dissipation a different τ' applied [11]. Border dissipativity in a BS evolution model should then be associated

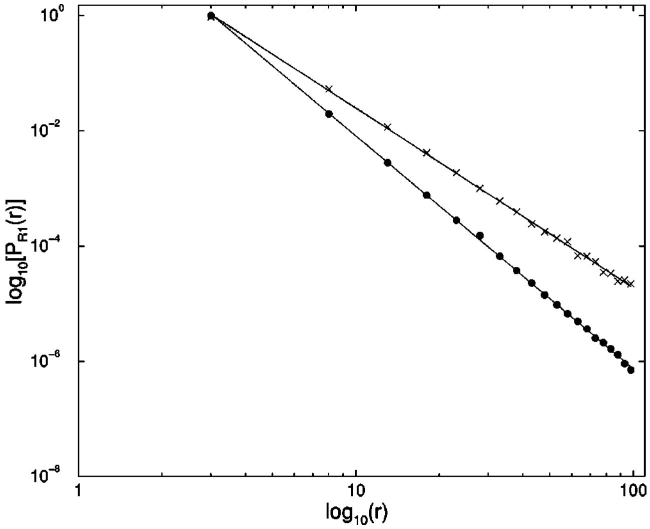

FIG. 2. Log-log plots of $P_{R1}(r)$ for b.c. (2) (dots) and b.c. (3) (crosses).

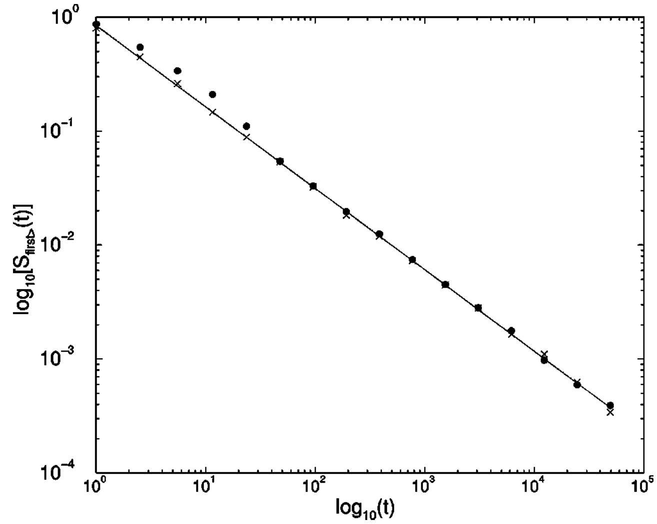

FIG. 3. Log-log plots of the integrated first return time distributions in the case of b.c. (2) (dots) and b.c. (3) (crosses).

with the fact that the average number of sites activated by the extremal site is less than the critical bulk value.

Figure 2 illustrates a numerical determination of τ_R' for avalanches starting with dissipative b.c. (2). We obtain $\tau_R' = 4.05 \pm 0.05$ in good agreement with our exact result. With b.c. (3) we get $\tau_R' = 3.02 \pm 0.08$, compatible in this case with the exact $\tau_R' = \tau_R = 3$ (Fig. 2).

III. TIME FRACTAL PROPERTIES

Avalanches of a BS model possess also time fractal properties, revealed, e.g., by the distribution of the first return times of the activity in a given site (time being measured by the number of minima which are replaced during the avalanche). Some general relations among exponents connected with the time and space fractal behavior in the bulk [6] can be easily derived by arguing as follows.

If we define as $S_{first}(t)$ the probability distribution of first return times in a given site, and call $n(T)$ the total number of returns in a lapse of time T, we expect $n(T) \propto T^{\tilde{d}}$, where $\tilde{d}$ is a time fractal dimension, and

$$\frac{T}{n(T)} \propto \int_1^T S_{first}(t) t dt. \tag{10}$$

Upon putting $S_{first}(t) \propto t^{-\tau_{first}}$, we get $\tilde{d} = \tau_{first} - 1$. Let us then call $S_{all}(t) \propto t^{-\tau_{all}}$ the distribution of times for all subsequent returns in a given site. We clearly have $\int_1^T S_{all}(t) dt \propto T^{\tilde{d}}$, so that $\tilde{d} = 1 - \tau_{all}$ and $\tau_{first} + \tau_{all} = 2$. At this point, to link space and time fractal properties it is sufficient to consider an avalanche as made of a total of s activated sites within a d-dimensional hyperspherical region of radius r such that $s \propto r^D$. If the avalanche has time duration t, we must have $s \propto r^d n(t) = r^d n(r^z)$, where z is an exponent connecting space and time ($t \propto r^z$). The last relation treats all lattice sites within the sphere as equivalent, as far as the return of activity is concerned. Now, since in our model $s = t$ by definition, $z = D$ also applies. Eventually, one finds

$$\tau_{first} = 2 - \frac{d}{D}. \tag{11}$$

This relation was first proposed in Ref. [6] for bulk exponents. The present derivation seems to be applicable also to avalanches starting at the border. The $d=1$, $M=\infty$ BS model is an ideal context in which to test its validity. In Ref. [10] a numerical estimate of τ_{first} was obtained which turned out to be compatible with the value implied by relation (11) ($\tau_{first} = 7/4$). We made a similar determination of τ_{first}' for avalanches starting at both dissipative and nondissipative borders. The data are plotted in Fig. 3, where one can clearly appreciate that the same values of τ_{first}' apply in the two cases. Indeed, for conservative b.c.'s [Eq. (3)] we estimate $\tau_{first}' = 1.71 \pm 0.03 \simeq 7/4$. With dissipative b.c.'s there appears to be a longer transient before the asymptotic time scaling behavior is established. However, we estimated $\tau_{first}' = 1.71 \pm 0.04$, clearly compatible again with 7/4. In both cases, of course, the bulk avalanches have a distribution with $\tau_{first} \simeq 7/4$.

Our results indicate that, like the space fractal dimension D', the time fractal dimension $\tilde{d}'$ of boundary avalanches is also the same as its bulk counterpart, with all b.c.'s, and that Eq. (11) is always satisfied.

In Ref. [8] a simulation of the $d=1$, $M=1$ BS model yielded a τ_{first}' sensibly different from τ_{first}. If such a discrepancy were confirmed by more systematic and asymptotic determinations, one should suspect that the identity of D' and D, or even the validity of some scaling relations like Eq. (11), is somehow granted here by the peculiar, classical character of the $M=\infty$ model. Anyhow, in such a case, a more complex scaling scenario would certainly apply to the model of Ref. [8].

IV. CONCLUSIONS

The M-component BS model in the limit $M \to \infty$ is an interesting theoretical laboratory for testing properties of the SOC state. With the present work we were able to compute analytically in $d=1$ the exponents τ' and τ_R' referring, respectively, to the size distribution and space correlation of avalanches starting at a border specified by b.c. (2). This extends the previous results in Ref. [10], which referred exclusively to bulk properties. In addition our formulation allowed a direct link between this model and the inhomogeneous branching process discussed in Ref. [8] to be established.

The results $\tau' = 7/4$ and $\tau'_R = 4$ show that the space fractal dimension D' of border avalanches with b.c. (2) remains equal to the bulk one $(D' = D = 4)$, in spite of the change of these exponents. Complemented by numerical tests, these results showed the existence of a clear-cut distinction between the b.c. in Eq. (2) and those expressed by Eq. (3). In analogy with the physics of sandpile models, we were led to call b.c. (2) dissipative, due to the fact that, in force of them, the boundary site, on average, is able to transmit activity to less than one site, even if the bulk is critical. This dissipativity is responsible for boundary values of the exponents τ' and τ'_R different from the bulk ones. On the other hand, when b.c.'s are conservative in the sense specified by Eq. (3), the existence of a geometrical boundary is not sufficient to determine a different scaling from the bulk.

Border dissipation, which for models like sandpiles is a necessary condition for the very establishment of the stationary SOC state, could be given here a precise meaning also in the context of a model with extremal dynamics. In this model dissipation reveals an essential ingredient for the existence of a peculiar boundary scaling, distinct from the bulk one. Indications that this could be a general feature of the SOC state come also from previous numerical results for sandpiles [11].

Our study of the return of activity at the border site revealed that dissipativity does not determine a new boundary τ'_{first} exponent, consistent with Eq. (11) and with the fact that, like D', for boundary avalanches also $\tilde{d}'$ remains unaltered with respect to its bulk value.

This contrasts with the numerical result $\tau'_{first} \neq \tau_{first}$ obtained in Ref. [8] for $M = 1$. Such a result, if confirmed by further analysis, awaits to be elucidated.

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